Available online at www.sciencedirect.com

SCIENCE DIRECT®

Advances in Applied Mathematics 34 (2005) 316–334

ADVANCES IN Applied Mathematics

www.elsevier.com/locate/yaama

The diagonal polynomials of dimension four

Hans L. Fetter [a], Juan H. Arredondo R. [a], Luis B. Morales [b,*]

[a] *Departamento de Matemáticas, Universidad Autónoma Metropolitana-I, México, D.F., 09340, Mexico*
[b] *IIMAS, Universidad Nacional Autónoma de México, México, D.F., 04510, Mexico*

Received 6 May 2004; accepted 10 August 2004

Available online 11 November 2004

Abstract

The characterization of all bijective polynomials from $\mathbb{N}^n$ to $\mathbb{N}$ (packing polynomials of dimension n) is a difficult unsolved problem. Apparently a more tractable problem is the determination of diagonal polynomials, a subset of packing polynomials. However for this later problem, it is only known that dimension two admits just one normalized diagonal polynomial (precisely the Cantor polynomial), and dimension three admits just two. Here, we prove that dimension four admits six normalized diagonal polynomials (normalized polynomials determine all diagonal polynomials).

1. Introduction

In this paper, $\mathbb{N}$ and $\mathbb{R}$ denote the nonnegative integers and real numbers. For $0 < n \in \mathbb{N}$ and $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{N}^n$, let $s(\mathbf{x}) = x_1 + \cdots + x_n$. Given any function f from $\mathbb{R}^n$ into $\mathbb{R}$ and any subset S of $\mathbb{R}^n$, write $f|S$ for the restriction of f to S. A map $f:\mathbb{R}^n \to \mathbb{R}$ is called a *packing function* of *dimension* n if the restriction $f|\mathbb{N}^n$ is a bijection onto $\mathbb{N}$. A function f is called a *diagonal function* if it is a packing function and $f(\mathbf{x}) < f(\mathbf{y})$ when $s(\mathbf{x}) < s(\mathbf{y})$ and $\mathbf{x}, \mathbf{y}$ in $\mathbb{N}^n$. Let $DF(n)$ and $DP(n)$ be the sets of n-dimensional diagonal *functions* and diagonal *polynomials*, respectively. Given any permutation π on $\{1, \ldots, n\}$

* Corresponding author.
E-mail addresses: hans@xanum.uam.mx (H.L. Fetter), iva@xanum.uam.mx (J.H. Arredondo R.), lbm@servidor.unam.mx (L.B. Morales).
URL: http://www.mcc.unam.mx (L.B. Morales).

0196-8858/$ – see front matter
doi:10.1016/j.aam.2004.08.004

and n-tuple $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{N}^n$, define $\pi\mathbf{x} = (x_{\pi(1)}, \ldots, x_{\pi(n)})$. Then, we say that two functions f and g on $\mathbb{R}^n$ are *equivalent* if there exists a permutation π such that for all $\mathbf{x}$, $f(\mathbf{x}) = g(\pi\mathbf{x})$. It is not hard to see that if $\mathbf{x} = (x_1, \ldots, x_n)$ is a variable vector and $0 < k \in \mathbb{N}$, then the binomial coefficient $\binom{k+s(\mathbf{x})}{k}$ produces a kth degree polynomial of the variables $x_1, \ldots, x_n$.

Any n-dimensional packing function f maps arbitrarily large n-dimensional arrays into computer memory cells numbered $0, 1, \ldots,$ and produces no conflicts in that process (see [11,12]). Moreover, f with polynomial formulas permits rapid assignment of memory location. This motivates that one tries to characterize the complete family of packing polynomials of dimension $n > 1$.

The characterization of all packing functions of dimension $n > 1$ is an old and very difficult problem. In his work on set theory, Cantor [1,2] gave for dimension two the diagonal polynomial

$$f(x_1, x_2) = \binom{x_1 + x_2 + 1}{2} + x_2$$

(he used this function to prove that $\mathbb{N}^2$ is a denumerable set). Fueter and Pólya [3] conjectured that, up to equivalence, f is the only packing polynomial of dimension two, and proved that it is the unique quadratic packing polynomial. Recently, Vsemirnov [15] gave two elementary proofs of this result. Later, Pólya and Szegö [10] excluded all polynomials that have the highest-degree homogeneous part vanishing nowhere, on the first real quadrant. Lew and Rosenberg [4] extended the Fueter–Pólya result. The Lew–Rosenberg results do not prove the uniqueness of f; they exclude all other polynomials of degree less than five. (See [14] for more information.) Apparently a more tractable problem is the determination of diagonal polynomials. In several papers [5,7–9,13] interest has centered on the description of diagonal polynomials. Recently Lew, Morales and Sánchez [6] established the fact that, up to equivalence, the Cantor polynomial is the only diagonal polynomial of dimension two. Moreover, they proved that, up to equivalence, there exist only two diagonal polynomials of dimension three:

$$g_1(x_1, x_2, x_3) = \binom{x_1 + x_2 + x_3 + 2}{3} + f(x_1, x_2),$$
$$g_2(x_1, x_2, x_3) = \binom{x_1 + x_2 + x_3 + 3}{3} - f(x_3, x_2) - 1.$$

The purpose of the present paper is to show that, up to equivalence, there are only six diagonal polynomials of dimension four, which were given in [8]. Here, they are determined using a combination of theoretical and computational results.

These results in dimensions two, three and four reinforce the conjecture that, up to equivalence, there are exactly $(n-1)!$ diagonal polynomials of dimension n (see [13]).

For $1 \leqslant i \leqslant n$, let $\mathbf{e}_i$ be the vector in $\mathbb{R}^n$ with ith component 1 and all the other components 0. By definition, if $f \in DP(n)$, then $f(0, \ldots, 0) = 0$, while $f(\pi\mathbf{e}_i) = i$ $(1 \leqslant i \leqslant n)$ for some permutation π. Call f *normalized* if $f(\mathbf{e}_i) = i$ for $i = 1, \ldots, n$. Let $NDP(n)$ be the set of normalized diagonal polynomials of dimension n. By definition, each f in $DP(n)$

determines a unique equivalent g in $NDP(n)$, so we need describe only the set $NDP(n)$. Given $w \in \mathbb{N}$, let

$$E(n,w) = \{\mathbf{x} \in \mathbb{N}^n \mid s(\mathbf{x}) = w\},$$
$$D(n,w) = \{\mathbf{x} \in \mathbb{N}^n \mid s(\mathbf{x}) \leqslant w\}.$$

If S is a set, we denote by $|S|$ the cardinality of S. It is not hard to prove the following relations:

(a) For all $\mathbf{x} \in \mathbb{N}^n$ $(n > 0)$, $|E(n, s(\mathbf{x}))| = \binom{n-1+s(\mathbf{x})}{n-1}$ and $|D(n, s(\mathbf{x}))| = \binom{n+s(\mathbf{x})}{n}$.
(b) If f is a diagonal function then it is a bijection from $D(n,w)$ onto $\{0, 1, \ldots, -1 + \binom{n+w}{n}\}$ for $w = 0, 1, \ldots$. Also, f is a bijection from $E(n,w)$ onto $\{\binom{n+w-1}{n}, \ldots, -1 + \binom{n+w}{n}\}$ for $w = 0, 1, \ldots$.

2. Preliminaries

In this section we use combinatorial arguments to obtain preliminary results.

If $f \in DP(n)$, then by [6, Theorem 2.3], f can be expressed as

$$f(\mathbf{x}) = \binom{n + s(\mathbf{x}) - 1}{n} + h(\mathbf{x}), \tag{1}$$

where h is a polynomial of total degree less than n. The polynomial h is called the *residue* of f. Also, for any such f we define

$$Rf(\mathbf{x}) = \binom{n + s(\mathbf{x}) - 1}{n} + \binom{n + s(\mathbf{x}) - 1}{n-1} - h(x_n, \ldots, x_1) - 1.$$

The following two Lemmas were proved in [6].

Lemma 1. *For each $w \in \mathbb{N}$, the residue h maps $E(n,w)$ bijectively onto*

$$\left\{0, 1, \ldots, -1 + \binom{n + w - 1}{n-1}\right\}.$$

Lemma 2. *If f in $DP(n)$ (respectively $NDP(n)$), then we have Rf in $DP(n)$ (respectively $NDP(n)$). Also $R(Rf) = f$.*

It follows from Lemma 1 that

$$0 \leqslant h(\mathbf{x}) \leqslant \binom{n + s(\mathbf{x}) - 1}{n-1} - 1. \tag{2}$$

Let f be a diagonal polynomial of dimension four and let h be its residue. Since the total degree of h is less than four and $h(0,\dots,0)=0$, the polynomial h can be written as

$$h(x_1,x_2,x_3,x_4)=\sum_{0<i+j+k+\ell\leqslant 3} b_{ijk\ell}x_1^i x_2^j x_3^k x_4^\ell, \quad \text{where } i,j,k,\ell\in\mathbb{N}.$$

However, we will find it more convenient to rewrite the polynomial h as follows:

$$h(x_1,x_2,x_3,x_4)=\sum_{0<i+j+k+\ell\leqslant 3} a_{ijk\ell}\binom{x_1}{i}\binom{x_2}{j}\binom{x_3}{k}\binom{x_4}{\ell} \tag{3}$$

for some constants $a_{ijk\ell}$ $(0\leqslant i,j,k,\ell\leqslant 3)$. Remember that $\binom{x}{i}:=x(x-1)\cdots(x-i+1)/i!$ is a polynomial in x. Observe that in the vector space of polynomials in four variables of degree less or equal to three this representation is unique, since the polynomials $\binom{x_1}{i},\dots,\binom{x_4}{\ell}$ form a basis. It follows immediately from this representation that all coefficients $a_{ijk\ell}$ of h must be integers, since they are really only differences of values of $h(x_1,x_2,x_3,x_4)$. Moreover these coefficients are bounded and we shall now proceed to approximate them as accurately as possible.

Lemma 3. *Let $f\in DP(4)$ and let h be its residue. If h has the form* (3)*, then*

(a) *For $1\leqslant i+j+k+l\leqslant 2$, we have $\alpha\leqslant a_{ijk\ell}\leqslant\beta \Leftrightarrow 3-\beta\leqslant a_{\ell kji}\leqslant 3-\alpha$.*
(b) *For $i+j+k+l=3$, we have $\alpha\leqslant a_{ijk\ell}\leqslant\beta \Leftrightarrow 1-\beta\leqslant a_{\ell kji}\leqslant 1-\alpha$.*

Proof. Let h' be the residue of Rf. Then the polynomial h' can be written as follows

$$\begin{aligned}
&h'(x_1,x_2,x_3,x_4)\\
&\quad=3\left[\binom{x_1}{1}+\cdots+\binom{x_4}{1}\right]+3\left[\binom{x_1}{2}+\cdots+\binom{x_4}{2}+\binom{x_1}{1}\binom{x_2}{1}+\cdots\right]\\
&\qquad+\left[\binom{x_1}{3}+\cdots+\binom{x_4}{3}+\binom{x_1}{2}\binom{x_2}{1}+\cdots+\binom{x_1}{1}\binom{x_2}{1}\binom{x_3}{1}+\cdots\right]\\
&\qquad-h(x_4,x_3,x_2,x_1)\\
&\quad=\sum_{1\leqslant i+j+k+\ell\leqslant 2}(3-a_{\ell kji})\binom{x_1}{i}\binom{x_2}{j}\binom{x_3}{k}\binom{x_4}{\ell}\\
&\qquad+\sum_{i+j+k+\ell=3}(1-a_{\ell kji})\binom{x_1}{i}\binom{x_2}{j}\binom{x_3}{k}\binom{x_4}{\ell}.
\end{aligned}$$

This implies the lemma because Rf is a diagonal polynomial (see Lemma 2). □

3. Bounds for the coefficients of the residue *h*

In this section we find by combinatorial arguments some bounds for the coefficients of the residue of any diagonal polynomial. Here, f denotes a normalized diagonal polynomial with residue h. Moreover, we assume that the polynomial h has the form (3) with coefficients $a_{ijk\ell}$.

Lemma 4. *For $i = 1, \dots, 4$, the value of $a_{3\mathbf{e}_i}$ is either 0 or 1.*

Proof. Evaluating h at points $w\mathbf{e}_i$ for $i = 1, \dots, 4$, we obtain, by applying (2),

$$0 \leqslant \frac{1}{6} a_{3\mathbf{e}_i} w^3 + \text{lower order terms} \leqslant \frac{1}{6} w^3 + \text{lower order terms}.$$

Since these inequalities hold for any value $w \in \mathbb{N}$ and $a_{3\mathbf{e}_i}$ is an integer, it follows that $a_{3\mathbf{e}_i} \in \{0, 1\}$. □

Lemma 5. *For $i = 1, \dots, 4$, we have $a_{\mathbf{e}_i} = i - 1$.*

Proof. Using (1) we get the result. □

Lemma 6.

$$\begin{aligned} 0 &\leqslant a_{2000} \leqslant 5, \\ -1 &\leqslant a_{0200} \leqslant 5, \\ -2 &\leqslant a_{0020} \leqslant 4, \\ -2 &\leqslant a_{0002} \leqslant 3. \end{aligned}$$

Proof. From (2) we obtain $0 \leqslant h(4\mathbf{e}_i) \leqslant 34$. It follows that $0 \leqslant 4(i-1) + 6a_{2\mathbf{e}_i} + 4a_{3\mathbf{e}_i} \leqslant 34$, hence

$$-\frac{1}{6}\left[4(i-1) + 4a_{3\mathbf{e}_i}\right] \leqslant a_{2\mathbf{e}_i} \leqslant \frac{1}{6}\left[34 - 4(i-1) - 4a_{3\mathbf{e}_i}\right].$$

Since $a_{3\mathbf{e}_i} \in \{0, 1\}$, we have

$$-\frac{1}{6}\left[4(i-1) + 4\right] \leqslant a_{2\mathbf{e}_i} \leqslant \frac{1}{6}\left[34 - 4(i-1)\right].$$

This inequality proves the lemma, because $a_{2\mathbf{e}_i}$ is an integer. □

Lemma 7. *For $1 \leqslant i \neq j \leqslant 4$ we have $2 - i - j \leqslant a_{\mathbf{e}_i + \mathbf{e}_j} \leqslant 11 - i - j$.*

Proof. By (2) we obtain $0 \leqslant h(\mathbf{e}_i + \mathbf{e}_j) \leqslant 9$. So $0 \leqslant i - 1 + a_{\mathbf{e}_i + \mathbf{e}_j} + j - 1 \leqslant 9$. This implies the lemma. □

Table 1
Possible values for $(a_{\mathbf{e}_i+2\mathbf{e}_j}, a_{2\mathbf{e}_i+\mathbf{e}_j})$, $i \neq j$

$a_{\mathbf{e}_i+2\mathbf{e}_j}$	−1	1	2	0	−2	3	−1	2	0	0	1	1
$a_{2\mathbf{e}_i+\mathbf{e}_j}$	1	−1	0	2	3	−2	2	−1	0	1	0	1

Lemma 8. *For distinct i and j, the coefficients $a_{\mathbf{e}_i+2\mathbf{e}_j}$ and $a_{2\mathbf{e}_i+\mathbf{e}_j}$ can take only the pairs of values given in Table 1.*

Proof. By (2) we obtain

$$0 \leqslant h(w\mathbf{e}_i + w\mathbf{e}_j) \leqslant \binom{2w+3}{3} - 1$$

for $w \in \mathbb{N}$. It follows that

$$0 \leqslant \left(\frac{1}{6}[a_{3\mathbf{e}_i} + a_{3\mathbf{e}_j}] + \frac{1}{2}[a_{\mathbf{e}_i+2\mathbf{e}_j} + a_{2\mathbf{e}_i+\mathbf{e}_j}]\right)w^3 + \cdots \leqslant \frac{8}{6}w^3 + \cdots.$$

So, the leading coefficient must satisfy

$$0 \leqslant \frac{1}{6}[a_{3\mathbf{e}_i} + a_{3\mathbf{e}_j}] + \frac{1}{2}[a_{\mathbf{e}_i+2\mathbf{e}_j} + a_{2\mathbf{e}_i+\mathbf{e}_j}] \leqslant \frac{8}{6}.$$

Since $a_{3\mathbf{e}_i}$ and $a_{3\mathbf{e}_j}$ are 0 or 1, it follows that

$$0 \leqslant \frac{1}{6}[a_{3\mathbf{e}_i} + a_{3\mathbf{e}_j}] \leqslant \frac{2}{6},$$

hence

$$-\frac{2}{6} \leqslant \frac{1}{2}[a_{\mathbf{e}_i+2\mathbf{e}_j} + a_{2\mathbf{e}_i+\mathbf{e}_j}] \leqslant \frac{8}{6}, \quad \text{implying} \quad -\frac{4}{6} \leqslant a_{\mathbf{e}_i+2\mathbf{e}_j} + a_{2\mathbf{e}_i+\mathbf{e}_j} \leqslant \frac{16}{6}.$$

Since $a_{\mathbf{e}_i+2\mathbf{e}_j}$ and $a_{2\mathbf{e}_i+\mathbf{e}_j}$ are integers, we have

$$0 \leqslant a_{\mathbf{e}_i+2\mathbf{e}_j} + a_{2\mathbf{e}_i+\mathbf{e}_j} \leqslant 2. \tag{4}$$

Similarly, considering that

$$0 \leqslant h(2w\mathbf{e}_i + w\mathbf{e}_j) \leqslant \binom{3w+3}{3} - 1,$$
$$0 \leqslant h(w\mathbf{e}_i + 2w\mathbf{e}_j) \leqslant \binom{3w+3}{3} - 1,$$

we obtain the inequalities

$$-1 \leqslant a_{\mathbf{e}_i+2\mathbf{e}_j} + 2a_{2\mathbf{e}_i+\mathbf{e}_j} \leqslant 4, \tag{5}$$

$$-1 \leqslant 2a_{\mathbf{e}_i+2\mathbf{e}_j} + a_{2\mathbf{e}_i+\mathbf{e}_j} \leqslant 4. \tag{6}$$

The only possible integer values satisfying (4)–(6) are those shown in Table 1. □

Now we can determine the valid combinations for the coefficients of the higher order terms of h.

Lemma 9. *For distinct i and j we have*

(a) *If $a_{2\mathbf{e}_i+\mathbf{e}_j} \geqslant 2$ then $a_{3\mathbf{e}_i} = 0$.*
(b) *If $a_{2\mathbf{e}_i+\mathbf{e}_j} \leqslant -1$ then $a_{3\mathbf{e}_i} = 1$.*

Proof. From Lemma 3 it suffices to prove (a). By (2) we obtain

$$0 \leqslant h(3w\mathbf{e}_i + w\mathbf{e}_j) \leqslant \binom{4w+3}{3} - 1$$

for $w \in \mathbb{N}$. Hence

$$0 \leqslant \frac{1}{6}[27a_{3\mathbf{e}_i} + a_{3\mathbf{e}_j}] + \frac{1}{2}[9a_{2\mathbf{e}_i+\mathbf{e}_j} + 3a_{\mathbf{e}_i+2\mathbf{e}_j}] \leqslant \frac{64}{6},$$

so

$$0 \leqslant 27a_{3\mathbf{e}_i} + a_{3\mathbf{e}_j} + 27a_{2\mathbf{e}_i+\mathbf{e}_j} + 9a_{\mathbf{e}_i+2\mathbf{e}_j} \leqslant 64. \tag{7}$$

Since $a_{2\mathbf{e}_i+\mathbf{e}_j} \geqslant 2$ by hypothesis, Lemma 8 implies that $3a_{2\mathbf{e}_i+\mathbf{e}_j} + a_{\mathbf{e}_i+2\mathbf{e}_j} \geqslant 5$. Then

$$-27a_{2\mathbf{e}_i+\mathbf{e}_j} - 9a_{\mathbf{e}_i+2\mathbf{e}_j} \leqslant -45. \tag{8}$$

It follows from (7) and (8) that

$$27a_{3\mathbf{e}_i} + a_{3\mathbf{e}_j} \leqslant 19. \tag{9}$$

Since $a_{3\mathbf{e}_j} \geqslant 0$, (9) implies $a_{3\mathbf{e}_i} = 0$. □

Lemma 10. *For $1 \leqslant i \leqslant 4$ we have*

(a) *If $a_{3\mathbf{e}_i} = 1$ then $a_{2\mathbf{e}_i} \leqslant 3$.*
(b) *If $a_{3\mathbf{e}_i} = 0$ then $a_{2\mathbf{e}_i} \geqslant 0$.*

Proof. By Lemma 3, (a) ⇔ (b); so we only have to prove (a). Suppose $a_{3\mathbf{e}_i} = 1$. Then by (2) we obtain $(a_{2\mathbf{e}_i} - 1)w^2 \leqslant 2w^2$. This inequality holds for any $w \in \mathbb{N}$, hence we must have $a_{2\mathbf{e}_i} \leqslant 3$. □

Lemma 11. *For distinct i, j and k $(1 \leqslant i, j, k \leqslant 4)$ we have that $a_{\mathbf{e}_i+\mathbf{e}_j+\mathbf{e}_k}$ belongs to the interval* $[-3, 4]$.

Proof. By (2) we obtain

$$0 \leqslant h(w\mathbf{e}_i + w\mathbf{e}_j + w\mathbf{e}_k) \leqslant \binom{3w+3}{3} - 1$$

for any $w \in \mathbb{N}$. It follows that

$$\begin{aligned} 0 &\leqslant \frac{1}{6}[a_{3\mathbf{e}_i} + a_{3\mathbf{e}_j} + a_{3\mathbf{e}_k}] \\ &\quad + \frac{1}{2}[a_{2\mathbf{e}_i+\mathbf{e}_j} + a_{\mathbf{e}_i+2\mathbf{e}_j} + a_{2\mathbf{e}_i+\mathbf{e}_k} + a_{\mathbf{e}_i+2\mathbf{e}_k} + a_{2\mathbf{e}_j+\mathbf{e}_k} + a_{\mathbf{e}_j+2\mathbf{e}_k}] + a_{\mathbf{e}_i+\mathbf{e}_j+\mathbf{e}_k} \\ &\leqslant \frac{27}{6}. \end{aligned}$$

By Lemmas 4 and 8, $a_{3\mathbf{e}_l}$ is either 0 or 1, and $0 \leqslant a_{2\mathbf{e}_l+\mathbf{e}_p} + a_{\mathbf{e}_l+2\mathbf{e}_p} \leqslant 2$ for $1 \leqslant l \neq p \leqslant 4$. So the above inequality implies the lemma. □

4. Discarding processes

In this section we describe the computational work used to give more accurate bounds for the coefficients of the residue of any normalized diagonal polynomial.

Let $\mathcal{G}$ be the set of all polynomials of the form (3) whose coefficients $a_{ijk\ell}$ satisfy Lemmas 4–11. Clearly, by definition, $\mathcal{G}$ contains all residues. Using these lemmas we can prove that $\mathcal{G}$ is very large: $|\mathcal{G}| \approx 5.36 \times 10^{18}$. So the obvious direct exhaustive search is not a feasible method to find all normalized diagonal polynomials. However, we design a computational technique that is powerful enough to lead to the result.

We introduce some notation before stating our results.

Let $\mathbf{x}_1, \ldots, \mathbf{x}_q \in \mathbb{R}^n$; we define $\langle \mathbf{x}_1, \ldots, \mathbf{x}_q \rangle = \{d_1\mathbf{x}_1 + \cdots + d_q\mathbf{x}_q\colon d_1, \ldots, d_q \in \mathbb{N}\}$. Let $\mathcal{H} = \{h\colon h$ is the residue of a normalized diagonal polynomial of dimension four$\}$. Given $\mathcal{M} \subset \mathcal{G}$, nonempty subsets U_t of $\mathbb{N}^4$ $(t = 1, \ldots, r)$, $(i_s, j_s, k_s, \ell_s) \in \mathbb{N}^4$ and $c_s \in \mathbb{Z}$ $(s = 1, \ldots, p \leqslant 34)$. Let $\mathcal{Q}$ denote the set of all $g \in \mathcal{M}$ whose coefficients $a_{i_s j_s k_s \ell_s}$ are respectively c_s for $s = 1, \ldots, p$. We define recursively the sequence $\{\mathcal{P}_t\}_{t=1}^{r}$ of sets as follows:

$$\begin{aligned} V_1 &= U_1, \qquad V_t = V_{t-1} \cup U_t, \quad t = 2, \ldots, r, \\ \mathcal{P}_1 &= \bigl\{g|V_1\colon g \in \mathcal{Q},\ g|U_1 \text{ obeys (2) and is injective in } E(4, w) \text{ for } w \geqslant 2\bigr\}, \\ &\vdots \\ \mathcal{P}_r &= \bigl\{g|V_r\colon g \in \mathcal{Q},\ g|V_{r-1} \in \mathcal{P}_{r-1},\ g|U_r \text{ obeys (2) and is injective in } E(4, w) \text{ and if} \\ &\qquad (\mathbf{x}, \mathbf{y}) \in U_j \times U_r,\ 1 \leqslant j < r,\ \mathbf{x} \neq \mathbf{y},\ s(\mathbf{x}) = s(\mathbf{y}) \text{ then } g(\mathbf{x}) \neq g(\mathbf{y}) \text{ for } w \geqslant 2\bigr\}. \end{aligned}$$

It is not hard to prove that

$$\mathcal{P}_j = \big\{g|V_t\colon\ g \in \mathcal{Q},\ g|V_t \text{ satisfies (2) and is injective in } E(4,w) \text{ for } w \geqslant 2\big\},$$

for $t = 1, \dots, r$. Also it is not hard to see that the computational work to calculate $\mathcal{P}_t$ is much less than that needed for the latter defined set. This is the reason of our recursive definition.

Definition 12. $(\mathcal{M}, \{\mathcal{P}_t\}_{t=1}^r, \{U_t\}_{t=1}^r, \{(i_s, j_s, k_s, \ell_s)\}_{s=1}^p, \{c_s\}_{s=1}^p)$ is called a process with parameters $\mathcal{M}$, U_t $(t = 1, \dots, r)$ and (i_s, j_s, k_s, ℓ_s), c_s $(s = 1, \dots, p)$. If $\mathcal{P}_r = \emptyset$, we say that the process is *discarding*.

Henceforth, let $f \in NDP(4)$ and let h be its residue. We also assume that h has the form (3).

Lemma 13. *For any parameters $\mathcal{M}$, $(\mathcal{H} \subset \mathcal{M})$ (i_s, j_s, k_s, ℓ_s) and c_s $(s = 1, \dots, p)$, we can find nonempty subsets U_t of $\mathbb{N}^4$ $(t = 1, \dots, r)$ such that the process $\{\mathcal{P}_t\}_{t=1}^r$ is discarding if and only if $c_1, \dots, c_p$ are not the respective coefficients $a_{i_1 j_1 k_1 \ell_1} \dots, a_{i_p j_p k_p \ell_p}$ of a residue of a normalized diagonal polynomial.*

Proof. Suppose $c_1, \dots, c_p$ are respectively the coefficients $a_{i_1 j_1 k_1 \ell_1}, \dots, a_{i_p j_p k_p \ell_p}$ of the residue h of a normalized diagonal polynomial. Since $\mathcal{H} \subset \mathcal{M}$, h belongs to $\mathcal{M}$. Then from Lemmas 4–11, the polynomial $h|V_t$ belongs to $\mathcal{P}_t$ $(1 \leqslant t \leqslant r)$ for any process $\{\mathcal{P}_t\}_{t=1}^r$.

Conversely, let g be a polynomial in $\mathcal{M}$ with the coefficients $a_{i_s j_s k_s \ell_s} = c_s$ for $s = 1, \dots, p$. By hypothesis, g is not the residue of a normalized diagonal polynomial. It follows from Lemma 1 that there exists $w \geqslant 2$ such that g is not a bijection from $E(4,w)$ onto $\big\{0, 1, \dots, -1 + \binom{3+w}{3}\big\}$. Then, by definition, the process $(\mathcal{P}_1, \mathcal{G}, E(4,w))$ is discarding. $\square$

Since the coefficient a_{1000} of the residue of any diagonal polynomial is zero, the previous lemma implies the following corollary.

Corollary 14. *Let $\mathcal{M}$ be a subset of $\mathcal{G}$ containing $\mathcal{H}$. If $h \in \mathcal{H}$, then for any process $(\mathcal{M}, \{\mathcal{P}_t\}_{t=1}^r, \{U_t\}_{t=1}^r, \{(1,0,0,0)\}, \{0\})$, we have $h|V_t \in \mathcal{P}_t$ for $t = 1 \dots, r$.*

All discarding processes used in this paper were implemented in C language. They are available for free downloading from the website:

`http://www.mcc.unam.mx/lbm/software.html`

5. Possible values of the coefficients $a_{2\mathbf{e}_i+\mathbf{e}_j}$ $(i \neq j)$

Let f be a diagonal normalized polynomial of dimension four and let h be its residue. Suppose that h has the form (3). Here, we use some discarding processes to prove that for $i \neq j$, $a_{2\mathbf{e}_i+\mathbf{e}_j} \in \{0, 1\}$.

Lemma 15. *If $a_{2\mathbf{e}_i+\mathbf{e}_j}$ and $a_{\mathbf{e}_i+2\mathbf{e}_j}$ $(i \neq j)$ are the coefficients of h, then* $(a_{2\mathbf{e}_i+\mathbf{e}_j}, a_{\mathbf{e}_i+2\mathbf{e}_j}) \notin \{(0,2),(2,0),(1,-1),(-1,1)\}$.

Proof. It follows from Lemma 3 that

$$(a_{2\mathbf{e}_i+\mathbf{e}_j}, a_{\mathbf{e}_i+2\mathbf{e}_j}) \in \big\{(0,2),(2,0)\big\} \quad \Leftrightarrow \quad (a_{\mathbf{e}_j+2\mathbf{e}_i}, a_{2\mathbf{e}_j+\mathbf{e}_i}) \in \big\{(1,-1),(-1,1)\big\}.$$

Thus, it suffices to prove that $(a_{2\mathbf{e}_i+\mathbf{e}_j}, a_{\mathbf{e}_i+2\mathbf{e}_j}) \notin \{(0,2),(2,0)\}$. There are 6 ordered pairs $(a_{2\mathbf{e}_i+\mathbf{e}_j}, a_{\mathbf{e}_i+2\mathbf{e}_j})$ for $i \neq j$. Note that each ordered pair of values (i,j) provides two cases since $(2,0)$ and $(0,2)$ are different ones. So, there are 12 cases to consider.

Suppose first that $(a_{2100}, a_{1200}) = (0,2)$. We will prove that the process

$$\big(\mathcal{P}_1, \mathcal{G}, \langle \mathbf{e}_1, \mathbf{e}_2\rangle, (2,1,0,0), (1,2,0,0), \{0,2\}\big)$$

is discarding. In consequence, Lemma 13 shows that $(a_{2100}, a_{1200}) = (0,2)$ is not possible.

Let $V_1 = U_1 = \langle \mathbf{e}_1, \mathbf{e}_2\rangle$ and $g \in \mathcal{G}$. Then any polynomial $g|V_1$ in $\mathcal{P}_1$ has the form

$$\begin{aligned} g(x_1,x_2,0,0) = x_2 + a_{2000}\binom{x_1}{2} + a_{1100}x_1x_2 + a_{0200}\binom{x_2}{2} \\ + a_{0300}\binom{x_2}{3} + a_{3000}\binom{x_1}{3} + 2x_1\binom{x_2}{2}. \end{aligned}$$

Using Lemmas 6, 7, 9 and 10, we can calculate that the number of possible polynomials of the above form is 600. We evaluate each one of these 600 polynomials on the points of $D(4,36)$. 172 of these polynomials do not satisfy (2), and the remaining 428 polynomials are not injective in $E(4,w)$, at least for some $2 \leqslant w \leqslant 36$. Thus, $\mathcal{P}_1 = \emptyset$.

Now suppose that $(a_{2100}, a_{1200}) = (2,0)$. In this case, we will prove that the process

$$\big(\{\mathcal{P}_l\}_{l=1}^2, \mathcal{G}, \langle \mathbf{e}_1, \mathbf{e}_2\rangle, \langle \mathbf{e}_2, \mathbf{e}_4\rangle, (2,1,0,0), (1,2,0,0), \{2,0\}\big)$$

is discarding. Thus, from Lemma 13, $(a_{2100}, a_{1200}) \neq (2,0)$.

For this process we take $V_1 = U_1 = \langle \mathbf{e}_1, \mathbf{e}_2\rangle$, $U_2 = \langle \mathbf{e}_1, \mathbf{e}_2\rangle$. Let $g \in \mathcal{G}$. Using again Lemmas 6, 7, 9 and 10 we obtain now that the number of polynomials $g|V_1$ to be considered is 660. After evaluation of $g|V_1 \in \mathcal{P}_1$ on points in $D(4,36)$ condition (2) rejects 167 and it turns out that 491 of the remaining are not injective in $E(4,w)$ at least for some $2 \leqslant w \leqslant 36$. Hence, $\mathcal{P}_1$ has only two polynomials:

$$\begin{aligned} g_1(x_1,x_2,0,0) &= x_2 + 4\binom{x_1}{2} - x_1x_2 + \binom{x_2}{2} + 2\binom{x_1}{2}x_2, \\ g_2(x_1,x_2,0,0) &= x_2 + 5\binom{x_1}{2} + 2\binom{x_2}{2} + 2\binom{x_1}{2}x_2. \end{aligned}$$

Let $V_2 = V_1 \cup U_2$. Then the polynomials $g|V_2$ on U_2 such that $g|V_1 \in \mathcal{P}_1$ have either the form

Table 2
Discarding processes for Lemma 15

(a_{2100}, a_{1200})	l	1	2
$(0,2)$	U_l	$\langle \mathbf{e}_1, \mathbf{e}_2 \rangle$	–
	$\lvert\mathcal{P}_l\rvert$	0	–
$(2,0)$	U_l	$\langle \mathbf{e}_1, \mathbf{e}_2 \rangle$	$\langle \mathbf{e}_2, \mathbf{e}_4 \rangle$
	$\lvert\mathcal{P}_l\rvert$	2	0

$$g_1(0, x_2, 0, x_4) = x_2 + 3x_4 + \binom{x_2}{2} + a_{0101}x_2x_4 + a_{0002}\binom{x_4}{2} + a_{0003}\binom{x_4}{3} + a_{0102}x_2\binom{x_4}{2} + a_{0201}\binom{x_2}{2}x_4 \tag{10}$$

or

$$g_2(0, x_2, 0, x_4) = x_2 + 3x_4 + 2\binom{x_2}{2} + a_{0101}x_2x_4 + a_{0002}\binom{x_4}{2} + a_{0003}\binom{x_4}{3} + a_{0102}x_2\binom{x_4}{2} + a_{0201}\binom{x_2}{2}x_4. \tag{11}$$

Using again Lemmas 6, 7, 9 and 10 we can show that both numbers of polynomials given in (10) and (11) are 720. Each one of these 1440 $(= 720 + 720)$ polynomials is evaluated on points of $D(4, 36)$. 16 (respectively 37) polynomials of the form (10) (respectively (11)) do not satisfy (2). Evaluation of $g_1|V_2$ (respectively $g_2|V_2$) on points of $E(4, w) \cap (\langle \mathbf{e}_1, \mathbf{e}_2 \rangle \cup \langle \mathbf{e}_2, \mathbf{e}_4 \rangle)$ shows that each of the remaining 704 (respectively 683) functions are not injective in that set at least for some $2 \leqslant w \leqslant 36$. So, $\mathcal{P}_2 = \emptyset$.

Table 2 shows schematically the previous two discarding processes.

Hereafter, in order to save space, the discarding processes used to reject values for the coefficients of the polynomial residue will be displayed in tables. Moreover, the evaluations of the polynomials considered were on the sets $E(4, w)$, $w = 2, \ldots, 36$.

The discarding processes used to prove that $(a_{2\mathbf{e}_i+\mathbf{e}_j}, a_{\mathbf{e}_i+2\mathbf{e}_j}) \neq (2, 0)$ for the pairs $(i, j) = (1, 4), (2, 3), (3, 4)$ are given in Table 3. As a consequence, Lemma 13 implies that these values are not possible. Here, $\mathcal{M} = \mathcal{G}$.

The parameters used in the processes to discard the remaining seven cases were $\mathcal{M} = \mathcal{G}$ and $U_1 = \langle \mathbf{e}_i, \mathbf{e}_j \rangle$. This and Lemma 13 complete the proof. □

Lemma 16. *If $a_{2\mathbf{e}_i+\mathbf{e}_j}$ and $a_{\mathbf{e}_i+2\mathbf{e}_j}$ $(i \neq j)$ are coefficients of h, then $(a_{2\mathbf{e}_i+\mathbf{e}_j}, a_{\mathbf{e}_i+2\mathbf{e}_j}) \notin \{(3, -2), (-2, 3)\}$.*

Proof. The proof is similar to that of Lemma 15. Here the parameter $\mathcal{M}$ is a set of polynomials in $\mathcal{G}$ such that their coefficients satisfy also Lemma 15. According to this definition we have $\mathcal{H} \subset \mathcal{M}$. The processes used to prove that $(a_{2\mathbf{e}_i+\mathbf{e}_j}, a_{\mathbf{e}_i+2\mathbf{e}_j}) \notin \{(3, -2), (-2, 3)\}$ are given in Table 4.

Table 3
Discarding processes for Lemma 15

(a_{2001}, a_{1002})	l	1	2	3
(2, 0)	U_l	$\langle \mathbf{e}_1, \mathbf{e}_4 \rangle$	$\langle \mathbf{e}_4, \mathbf{e}_3 \rangle$	–
	$\lvert\mathcal{P}_l\rvert$	1	0	–
(a_{0210}, a_{0120})				
(2, 0)	U_l	$\langle \mathbf{e}_2, \mathbf{e}_3 \rangle$	$\langle \mathbf{e}_3, \mathbf{e}_4 \rangle$	–
	$\lvert\mathcal{P}_l\rvert$	3	0	–
(a_{0021}, a_{0012})				
(2, 0)	U_l	$\langle \mathbf{e}_3, \mathbf{e}_4 \rangle$	$\langle \mathbf{e}_4, \mathbf{e}_1 \rangle$	$\langle \mathbf{e}_1, \mathbf{e}_2 \rangle$
	$\lvert\mathcal{P}_l\rvert$	2	1	0

Table 4
Discarding processes for Lemma 16

(a_{2001}, a_{1002})	l	1	2	3	4
(3, −2)	U_l	$\langle \mathbf{e}_1, \mathbf{e}_4 \rangle$	$\langle \mathbf{e}_4, \mathbf{e}_3 \rangle$	$\langle \mathbf{e}_3, \mathbf{e}_2 \rangle$	$\langle \mathbf{e}_2, \mathbf{e}_1 \rangle$
	$\lvert\mathcal{P}_l\rvert$	4	4	6	0
(−2, 3)	U_l	$\langle \mathbf{e}_4, \mathbf{e}_1 \rangle$	–	–	–
	$\lvert\mathcal{P}_l\rvert$	0	–	–	–
(a_{2010}, a_{1020})					
(3, −2)	U_l	$\langle \mathbf{e}_1, \mathbf{e}_3 \rangle$	$\langle \mathbf{e}_3, \mathbf{e}_4 \rangle$	$\langle \mathbf{e}_4, \mathbf{e}_2 \rangle$	$\langle \mathbf{e}_2, \mathbf{e}_1 \rangle$
	$\lvert\mathcal{P}_l\rvert$	8	17	8	0
(−2, 3)	U_l	$\langle \mathbf{e}_3, \mathbf{e}_1 \rangle$	$\langle \mathbf{e}_1, \mathbf{e}_4 \rangle$	–	–
	$\lvert\mathcal{P}_l\rvert$	2	0	–	–
(a_{2100}, a_{1200})					
(3, −2)	U_l	$\langle \mathbf{e}_1, \mathbf{e}_2 \rangle$	$\langle \mathbf{e}_2, \mathbf{e}_4 \rangle$	$\langle \mathbf{e}_4, \mathbf{e}_3 \rangle$	$\langle \mathbf{e}_3, \mathbf{e}_1 \rangle$
	$\lvert\mathcal{P}_l\rvert$	8	12	6	0
(−2, 3)	U_l	$\langle \mathbf{e}_2, \mathbf{e}_1 \rangle$	$\langle \mathbf{e}_1, \mathbf{e}_4 \rangle$	–	–
	$\lvert\mathcal{P}_l\rvert$	2	0	–	–
(a_{0210}, a_{0120})					
(3, −2)	U_l	$\langle \mathbf{e}_2, \mathbf{e}_3 \rangle$	$\langle \mathbf{e}_3, \mathbf{e}_4 \rangle$	$\langle \mathbf{e}_4, \mathbf{e}_1 \rangle$	–
	$\lvert\mathcal{P}_l\rvert$	8	17	0	–
(−2, 3)	U_l	$\langle \mathbf{e}_3, \mathbf{e}_2 \rangle$	$\langle \mathbf{e}_2, \mathbf{e}_4 \rangle$	–	–
	$\lvert\mathcal{P}_l\rvert$	2	0	–	–

Note that the number of distinct cases for the equalities $(a_{\mathbf{e}_i+2\mathbf{e}_j}, a_{2\mathbf{e}_i+\mathbf{e}_j}) = (3, -2)$ and $(a_{\mathbf{e}_i+2\mathbf{e}_j}, a_{2\mathbf{e}_i+\mathbf{e}_j}) = (-2, 3)$, $(i \neq j)$ is 12. However, Table 4 contains 8 of such possible cases. Moreover, Lemma 3 implies that

$$(a_{2100}, a_{1200}) \in \big\{(3, -2), (-2, 3)\big\} \quad \Leftrightarrow \quad (a_{0021}, a_{0012}) \in \big\{(-2, 3), (3, -2)\big\},$$
$$(a_{2010}, a_{1020}) \in \big\{(3, -2), (-2, 3)\big\} \quad \Leftrightarrow \quad (a_{0102}, a_{0201}) \in \big\{(-2, 3), (3, -2)\big\}.$$

Table 5
Discarding processes for Lemma 17

(a_{2001}, a_{1002})	l	1	2	3	4
$(2, -1)$	U_l	$\langle \mathbf{e}_1, \mathbf{e}_4 \rangle$	$\langle \mathbf{e}_4, \mathbf{e}_3 \rangle$	$\langle \mathbf{e}_3, \mathbf{e}_2 \rangle$	$\langle \mathbf{e}_2, \mathbf{e}_1 \rangle$
	$\lvert \mathcal{P}_l \rvert$	30	61	88	0
$(-1, 2)$	U_l	$\langle \mathbf{e}_4, \mathbf{e}_1 \rangle$	$\langle \mathbf{e}_1, \mathbf{e}_3 \rangle$	$\langle \mathbf{e}_3, \mathbf{e}_2 \rangle$	–
	$\lvert \mathcal{P}_l \rvert$	12	1	0	–
(a_{2010}, a_{1020})					
$(2, -1)$	U_l	$\langle \mathbf{e}_1, \mathbf{e}_3 \rangle$	$\langle \mathbf{e}_3, \mathbf{e}_4 \rangle$	$\langle \mathbf{e}_4, \mathbf{e}_2 \rangle$	$\langle \mathbf{e}_2, \mathbf{e}_1 \rangle$
	$\lvert \mathcal{P}_l \rvert$	25	56	10	0
$(-1, 2)$	U_l	$\langle \mathbf{e}_3, \mathbf{e}_1 \rangle$	$\langle \mathbf{e}_1, \mathbf{e}_4 \rangle$	$\langle \mathbf{e}_4, \mathbf{e}_2 \rangle$	$\langle \mathbf{e}_2, \mathbf{e}_3 \rangle$
	$\lvert \mathcal{P}_l \rvert$	15	9	2	0
(a_{2100}, a_{1200})					
$(2, -1)$	U_l	$\langle \mathbf{e}_1, \mathbf{e}_2 \rangle$	$\langle \mathbf{e}_2, \mathbf{e}_4 \rangle$	$\langle \mathbf{e}_4, \mathbf{e}_3 \rangle$	$\langle \mathbf{e}_3, \mathbf{e}_1 \rangle$
	$\lvert \mathcal{P}_l \rvert$	17	38	45	0
$(-1, 2)$	U_l	$\langle \mathbf{e}_2, \mathbf{e}_1 \rangle$	$\langle \mathbf{e}_1, \mathbf{e}_4 \rangle$	$\langle \mathbf{e}_4, \mathbf{e}_3 \rangle$	$\langle \mathbf{e}_3, \mathbf{e}_2 \rangle$
	$\lvert \mathcal{P}_l \rvert$	12	10	3	0
(a_{0210}, a_{0120})					
$(2, -1)$	U_l	$\langle \mathbf{e}_2, \mathbf{e}_3 \rangle$	$\langle \mathbf{e}_3, \mathbf{e}_4 \rangle$	$\langle \mathbf{e}_4, \mathbf{e}_1 \rangle$	$\langle \mathbf{e}_1, \mathbf{e}_2 \rangle$
	$\lvert \mathcal{P}_l \rvert$	14	41	2	0
$(-1, 2)$	U_l	$\langle \mathbf{e}_3, \mathbf{e}_2 \rangle$	$\langle \mathbf{e}_2, \mathbf{e}_4 \rangle$	$\langle \mathbf{e}_4, \mathbf{e}_1 \rangle$	$\langle \mathbf{e}_1, \mathbf{e}_3 \rangle$
	$\lvert \mathcal{P}_l \rvert$	12	7	0	–

Thus the other 4 cases are not valid. Therefore, Lemma 13 shows that the pairs $(3, -2)$ and $(-2, 3)$ for $(a_{\mathbf{e}_i+2\mathbf{e}_j}, a_{2\mathbf{e}_i+\mathbf{e}_j})$ $(i \neq j)$ are not possible. $\square$

Lemma 17. *If $a_{2\mathbf{e}_i+\mathbf{e}_j}$ and $a_{\mathbf{e}_i+2\mathbf{e}_j}$ $(i \neq j)$ are coefficients of h, then $(a_{2\mathbf{e}_i+\mathbf{e}_j}, a_{\mathbf{e}_i+2\mathbf{e}_j}) \notin \{(2, -1), (-1, 2)\}$.*

Proof. The proof is similar to that of Lemma 16. Here $\mathcal{M}$ is a set of polynomials in $\mathcal{G}$ such that their coefficients satisfy also Lemmas 15 and 16. According to this definition we have $\mathcal{H} \subset \mathcal{M}$. The processes used to prove that the pairs $(2, -1)$ and $(-1, 2)$ for $(a_{2\mathbf{e}_i+\mathbf{e}_j}, a_{\mathbf{e}_i+2\mathbf{e}_j})$ $(i \neq j)$ are not possible are given in Table 5.

Note that the number of all distinct cases for equations $(a_{2\mathbf{e}_i+\mathbf{e}_j}, a_{\mathbf{e}_i+2\mathbf{e}_j}) = (2, -1)$ and $(a_{2\mathbf{e}_i+\mathbf{e}_j}, a_{\mathbf{e}_i+2\mathbf{e}_j}) = (-1, 2)$ $(i \neq j)$, is 12. However, Table 5 contains eight of these possible cases. The other four cases are rejected using Table 5 and Lemma 3. Hence, the pairs $(2, -1)$ and $(-1, 2)$ for $(a_{2\mathbf{e}_i+\mathbf{e}_j}, a_{\mathbf{e}_i+2\mathbf{e}_j})$ $(i \neq j)$ are not possible. $\square$

As an immediate consequence of Lemmas 8 and 15–17 we have the following:

Corollary 18. *If $a_{2\mathbf{e}_i+\mathbf{e}_j}$ $(i \neq j)$ is a coefficient of h, then it is either* 0 *or* 1.

6. Discarding other values of the coefficients of *h*

In this section, $\mathcal{G}_1$ will denote the set of all polynomials in $\mathcal{G}$ such that their coefficients satisfy Corollary 18. This definition implies that $\mathcal{H} \subset \mathcal{G}_1$. Here, we continue to reject some of the values for the coefficients of h determined in Section 3.

Lemma 19. *For $i = 1, \ldots, 4$, the coefficient $a_{2\mathbf{e}_i}$ of h belongs to the interval* $[0, 3]$.

Proof. To prove that $a_{2000} \leqslant 3$ we take $\mathcal{M} = \mathcal{G}_1$ and proceed along the same lines as in Lemma 15. The second, third, fourth and fifth rows of Table 6 give the discarding processes used for this case.

Let $\mathcal{G}'_1$ be the set of all polynomials in $\mathcal{G}_1$ such that the coefficient a_{2000} is less or equal to 3. To prove that $0 \leqslant a_{0200} \leqslant 3$ we take $\mathcal{M} = \mathcal{G}'_1$. Rows 5–7 of Table 6 give the discarding processes used for this case.

By applying Lemma 3 we reject the corresponding values for a_{0002} and a_{0020}. So Lemma 6 completes the proof. □

Let $\mathcal{G}_2$ be the set of all polynomials in $\mathcal{G}_1$ such that their coefficients satisfy Lemma 19. It follows that $\mathcal{H} \subset \mathcal{G}_2$.

Lemma 20. *If $a_{\mathbf{e}_i+\mathbf{e}_j}$ $(i \neq j)$ is a coefficient of h, then*

(i) $-1 \leqslant a_{1100} \leqslant 3$,
(ii) $0 \leqslant a_{0011} \leqslant 4$,
(iii) $-2 \leqslant a_{1010} \leqslant 2$,
(iv) $1 \leqslant a_{0101} \leqslant 5$,
(v) $-1 \leqslant a_{1001} \leqslant 4$,
(vi) $1 \leqslant a_{0110} \leqslant 2$.

Table 6
Discarding processes for Lemma 19

a_{2000}	l	1	2	3	4
5	U_l	$\langle \mathbf{e}_1, \mathbf{e}_2 \rangle$	$\langle \mathbf{e}_2, \mathbf{e}_4 \rangle$	$\langle \mathbf{e}_4, \mathbf{e}_3 \rangle$	$\langle \mathbf{e}_3, \mathbf{e}_1 \rangle$
	$\lvert\mathcal{P}_l\rvert$	23	9	2	0
4	U_l	$\langle \mathbf{e}_1, \mathbf{e}_2 \rangle$	$\langle \mathbf{e}_2, \mathbf{e}_4 \rangle$	$\langle \mathbf{e}_4, \mathbf{e}_3 \rangle$	–
	$\lvert\mathcal{P}_l\rvert$	19	16	0	–
a_{0200}					
5	U_l	$\langle \mathbf{e}_2, \mathbf{e}_1 \rangle$	$\langle \mathbf{e}_1, \mathbf{e}_4 \rangle$	$\langle \mathbf{e}_4, \mathbf{e}_3 \rangle$	–
	$\lvert\mathcal{P}_l\rvert$	21	16	0	–
4	U_l	$\langle \mathbf{e}_2, \mathbf{e}_1 \rangle$	$\langle \mathbf{e}_1, \mathbf{e}_4 \rangle$	$\langle \mathbf{e}_4, \mathbf{e}_3 \rangle$	$\langle \mathbf{e}_3, \mathbf{e}_1 \rangle$
	$\lvert\mathcal{P}_l\rvert$	17	12	2	0
-1	U_l	$\langle \mathbf{e}_2, \mathbf{e}_1 \rangle$	$\langle \mathbf{e}_1, \mathbf{e}_4 \rangle$	$\langle \mathbf{e}_4, \mathbf{e}_3 \rangle$	$\langle \mathbf{e}_3, \mathbf{e}_2 \rangle$
	$\lvert\mathcal{P}_l\rvert$	9	19	4	0

Table 7
Discarding processes for Lemma 20(i)

a_{1100}	l	1	2	3	4
8	U_l	$\langle \mathbf{e}_1, \mathbf{e}_2 \rangle$	$\langle \mathbf{e}_2, \mathbf{e}_4 \rangle$	$\langle \mathbf{e}_4, \mathbf{e}_3 \rangle$	$\langle \mathbf{e}_3, \mathbf{e}_1 \rangle$
	$\lvert \mathcal{P}_l \rvert$	1	1	0	–
7	$\lvert \mathcal{P}_l \rvert$	5	5	0	–
6	$\lvert \mathcal{P}_l \rvert$	12	4	1	0
5	$\lvert \mathcal{P}_l \rvert$	16	4	1	0
4	$\lvert \mathcal{P}_l \rvert$	14	13	5	0

Table 8
Discarding processes for Lemma 20(iii)

a_{1010}	l	1	2	3	4
7	U_l	$\langle \mathbf{e}_1, \mathbf{e}_3 \rangle$	$\langle \mathbf{e}_3, \mathbf{e}_2 \rangle$	$\langle \mathbf{e}_2, \mathbf{e}_4 \rangle$	$\langle \mathbf{e}_4, \mathbf{e}_1 \rangle$
	$\lvert \mathcal{P}_l \rvert$	2	3	0	–
6	$\lvert \mathcal{P}_l \rvert$	16	23	11	0
5	$\lvert \mathcal{P}_l \rvert$	12	28	6	0
4	$\lvert \mathcal{P}_l \rvert$	18	22	6	0
3	$\lvert \mathcal{P}_l \rvert$	41	48	10	0

Proof. It follows from Lemma 3 that (i) $\Leftrightarrow$ (ii) and (iii) $\Leftrightarrow$ (iv). Thus, it suffices to prove (i), (iii), (v) and (vi). The parameters $\{U_l\}$ are shown once in the tables since they are the same in every case.

We first prove that $a_{1100} \leqslant 3$. Here $\mathcal{M} = \mathcal{G}_2$. Table 7 gives the discarding processes used in this case.

We now prove that $a_{1010} \leqslant 2$ Here again $\mathcal{M} = \mathcal{G}_2$. The discarding processes used in this case is given in Table 8.

Finally to prove $a_{1001} \leqslant 4$ and $a_{0110} \leqslant 2$, we take $\mathcal{M}$ as the set $\mathcal{G}_2'$ of polynomials in $\mathcal{G}_2$ such that $a_{\mathbf{e}_i+\mathbf{e}_j}$ $(i \neq j)$ satisfies (i), (ii), (iii) and (iv). Table 9 gives the discarding processes used in these cases.

To finish the proof one uses Lemma 7 in order to see that only the values stated for these coefficients are possible. $\square$

Lemma 21.

(i) $-1 \leqslant a_{1100} \leqslant 2$,
(ii) $1 \leqslant a_{0011} \leqslant 4$.

Proof. From Lemmas 3 and 20 it suffices to prove $a_{1100} \neq 3$. Here we take $\mathcal{M}$ as the set $\mathcal{G}_2''$ of polynomials in $\mathcal{G}_2$ such that their coefficients satisfy Lemma 20. The discarding processes used to prove this lemma are given in Table 10. $\square$

Lemma 22. *The coefficients a_{3000} and a_{0003} of h are respectively* 0 *and* 1.

Table 9
Discarding processes for Lemma 20(v)–(vi)

a_{1001}	l	1	2	3	4
6	U_l $\lvert\mathcal{P}_l\rvert$	$\langle \mathbf{e}_1, \mathbf{e}_4\rangle$ 6	$\langle \mathbf{e}_4, \mathbf{e}_2\rangle$ 2	$\langle \mathbf{e}_2, \mathbf{e}_3\rangle$ 0	$\langle \mathbf{e}_3, \mathbf{e}_1\rangle$ –
5	$\lvert\mathcal{P}_l\rvert$	11	17	2	0
a_{0110}	l	1	2	3	4
6	$\langle \mathbf{e}_2, \mathbf{e}_3\rangle$ $\lvert\mathcal{P}_l\rvert$	$\langle \mathbf{e}_3, \mathbf{e}_1\rangle$ 8	$\langle \mathbf{e}_1, \mathbf{e}_4\rangle$ 9	$\langle \mathbf{e}_4, \mathbf{e}_2\rangle$ 2	 0
5	$\lvert\mathcal{P}_l\rvert$	14	31	2	0
4	$\lvert\mathcal{P}_l\rvert$	14	69	3	0
3	$\lvert\mathcal{P}_l\rvert$	37	111	40	0

Table 10
Discarding processes for Lemma 21

a_{1100}	l	1	2	3	4
3	U_l $\lvert\mathcal{P}_l\rvert$	$\langle \mathbf{e}_1, \mathbf{e}_2\rangle$ 37	$\langle \mathbf{e}_2, \mathbf{e}_4\rangle$ 76	$\langle \mathbf{e}_4, \mathbf{e}_3\rangle$ 18	$\langle \mathbf{e}_3, \mathbf{e}_1\rangle$ 0

Table 11
Discarding processes for Lemma 22

(a_{3000}, a_{0003})	l	1	2	3	4	5
(1, 0)	U_l $\lvert\mathcal{P}_l\rvert$	$\langle \mathbf{e}_1, \mathbf{e}_4\rangle$ 14	$\langle \mathbf{e}_4, \mathbf{e}_2\rangle$ 3	$\langle \mathbf{e}_2, \mathbf{e}_3\rangle$ 0	$\langle \mathbf{e}_3, \mathbf{e}_1\rangle$ –	$\langle \mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_4\rangle$ –
(1,1)	$\lvert\mathcal{P}_l\rvert$	32	54	13	2	0

Proof. From Lemma 4 it suffices to prove that $(a_{3000}, a_{0003}) \notin \{(1, 0), (1, 1)\}$. Here the parameter $\mathcal{M}$ is the set $\mathcal{G}_2'''$ of all polynomials in $\mathcal{G}_2$ such that their coefficients satisfy Lemmas 20 and 21. It follows that $\mathcal{H} \subset \mathcal{M}$. The processes used in the lemma are given in Table 11. □

Let us define $\mathcal{G}_3$ as the set of all polynomials in $\mathcal{G}_2$ such that their coefficients satisfy Lemmas 20, 21 and 22. It follows that $\mathcal{H} \subset \mathcal{G}_3$.

Lemma 23. *For distinct i and j, the coefficient $a_{\mathbf{e}_i+\mathbf{e}_j}$ of h is nonnegative.*

Proof. From Lemma 20, it suffices to consider the following cases.

- (i) a_{1100}. For this case we have Table 12. Here $\mathcal{M} = \mathcal{G}_3$.
- (ii) a_{1010}. The previous case and Lemma 3 imply $a_{0011} < 4$. At this stage we take $\mathcal{M}$ as the set $\mathcal{G}_3'$ of all polynomials in $\mathcal{G}_3$ such that the coefficients satisfy $a_{1100} \geqslant 0$ and $a_{0011} \leqslant 3$. So we have Table 13.

Table 12
Discarding processes for Lemma 23(i)

a_{1100}	l	1	2	3	4	5
-1	U_l	$\langle \mathbf{e}_1, \mathbf{e}_2\rangle$	$\langle \mathbf{e}_2, \mathbf{e}_4\rangle$	$\langle \mathbf{e}_4, \mathbf{e}_3\rangle$	$\langle \mathbf{e}_3, \mathbf{e}_1\rangle$	$\langle \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_4\rangle$
	$\lvert\mathcal{P}_l\rvert$	4	58	58	4	0

Table 13
Discarding processes for Lemma 23(ii)

a_{1010}	l	1	2	3	4	5
-2	U_l	$\langle \mathbf{e}_1, \mathbf{e}_3\rangle$	$\langle \mathbf{e}_3, \mathbf{e}_2\rangle$	$\langle \mathbf{e}_2, \mathbf{e}_4\rangle$	$\langle \mathbf{e}_4, \mathbf{e}_1\rangle$	$\langle \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\rangle$
	$\lvert\mathcal{P}_l\rvert$	3	3	9	1	0
-1	$\lvert\mathcal{P}_l\rvert$	9	14	23	5	0

Table 14
Discarding processes for Lemma 23(iii)

a_{1001}	l	1	2	3	4
-1	U_l	$\langle \mathbf{e}_1, \mathbf{e}_4\rangle$	$\langle \mathbf{e}_4, \mathbf{e}_2\rangle$	$\langle \mathbf{e}_2, \mathbf{e}_3\rangle$	$\langle \mathbf{e}_3, \mathbf{e}_1\rangle$
	$\lvert\mathcal{P}_l\rvert$	7	15	8	0

(iii) a_{1001}. Now we take $\mathcal{M}$ as the set $\mathcal{G}_3''$ of polynomials in $\mathcal{G}_3'$ satisfying $a_{1010} \geqslant 0$ (ii). For this case we have Table 14.

Note that Table 14 implies $0 \leqslant a_{1001} \leqslant 3$. In all cases we have proved that $a_{\mathbf{e}_i+\mathbf{e}_j} \geqslant 0$ for $i \neq j$. $\square$

Lemma 24. *The coefficients of h satisfy*

$$a_{2000} = a_{1100} = 0 \quad \textit{and} \quad a_{0002} = a_{0011} = 3.$$

Proof. Since $h(\mathbf{e}_i + \mathbf{e}_j) = i - 1 + j - 1 + a_{\mathbf{e}_i+\mathbf{e}_j}$, the previous lemma states that $h(\mathbf{e}_i + \mathbf{e}_j) > 0$ for $i \neq j$. Clearly Lemma 19 implies $h(2\mathbf{e}_i) = 2(i-1) + a_{2\mathbf{e}_i} > 0$ for $i = 2, 3, 4$. By Lemma 1, h is a bijection from $E(4, 2)$ onto $\{0, \ldots, 9\}$, so $h(2\mathbf{e}_1) = a_{2000} = 0$; hence $h(2\mathbf{e}_4) = 9$ by (2) and Lemma 2. This shows that $a_{0002} = 3$.

From Lemma 23, $a_{\mathbf{e}_i+\mathbf{e}_j} \geqslant 0$ for $i \neq j$. Then $h(\mathbf{e}_i + \mathbf{e}_j) > 1$, unless $i = 1$ and $j = 2$. Hence $h(\mathbf{e}_1 + \mathbf{e}_2) = 1 + a_{1100} = 1$; so $a_{1100} = 0$. It follows from (2) and Lemma 2 that $a_{0011} = 3$. $\square$

7. All diagonal polynomials of dimension four

Let $\mathcal{G}_4$ be the set of all polynomials in $\mathcal{G}_3$ such that their coefficients satisfy Lemma 23 and 24. According to this definition we have $\mathcal{H} \subset \mathcal{G}_4$. Using these lemmas we can prove that $|\mathcal{G}_4| = 2, 516, 582, 420$. Since the set is still too big, direct searching to find all normalized diagonal polynomials is impractical. However, Corollary 14 will be used to determine the coefficients of any residue. So, we will prove that there exist only six normalized diagonal polynomials of dimension four.

Theorem 25. *If $f \in NDP(4)$, then its residue has only the forms*:

$$\begin{aligned} h_1(x_1,x_2,x_3,x_4) = {} & x_2+2x_3+3x_4+x_1x_3+x_2x_3+\binom{x_3}{2}+3x_1x_4 \\ & +3x_2x_4+3x_3x_4+3\binom{x_4}{2}+x_1x_2x_4+x_1x_3x_4+x_2x_3x_4+\binom{x_4}{3} \\ & +\binom{x_1}{2}x_4+\binom{x_2}{2}x_4+\binom{x_3}{2}x_4+x_1\binom{x_4}{2}+x_2\binom{x_4}{2}+x_3\binom{x_4}{2}, \end{aligned}$$

$$\begin{aligned} h_2(x_1,x_2,x_3,x_4) = {} & x_2+2x_3+3x_4+x_1x_3+x_2x_3+3\binom{x_3}{2}+2x_1x_4 \\ & +2x_2x_4+3x_3x_4+3\binom{x_4}{2}+\binom{x_3}{3}+x_1x_3x_4+x_2x_3x_4+\binom{x_4}{3} \\ & +x_1\binom{x_3}{2}+x_2\binom{x_3}{2}+\binom{x_3}{2}x_4+x_1\binom{x_4}{2}+x_2\binom{x_4}{2}+x_3\binom{x_4}{2}, \end{aligned}$$

$$\begin{aligned} h_3(x_1,x_2,x_3,x_4) = {} & x_2+2x_3+3x_4+\binom{x_2}{2}+x_2x_3+\binom{x_3}{2}+3x_1x_4 \\ & +3x_2x_4+3x_3x_4+3\binom{x_4}{2}+x_1x_2x_4+x_1x_3x_4+x_2x_3x_4+\binom{x_4}{3} \\ & +\binom{x_1}{2}x_4+\binom{x_2}{2}x_4+\binom{x_3}{2}x_4+x_1\binom{x_4}{2}+x_2\binom{x_4}{2}+x_3\binom{x_4}{2}, \end{aligned}$$

$$\begin{aligned} h_j(x_1,x_2,x_3,x_4) = {} & \binom{3+s(x_1,\dots,x_4)}{3} \\ & -h_{j-3}(x_4,\dots,x_1)-1, \quad \textit{for } j=4,5,6. \end{aligned}$$

Proof. The process considered here is given in Table 15. In this process $\mathcal{M}=\mathcal{G}_4$. Table 15 shows that $\mathcal{P}_6$ has six polynomials. Moreover, these six polynomials coincide with the above ones. Each function in $\mathcal{P}_6$ satisfies (3) and is injective in $E(4,w)$ for $w=2,\dots,36$. However, each one of these polynomials is precisely the residue of a normalized diagonal polynomial constructed in a previous paper [8]. It follows from Corollary 14 that $\mathcal{P}_6$ contains the residue of any diagonal polynomial because $V_6=\langle \mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_4\rangle=\mathbb{N}^4$. This completes the proof. □

Table 15
Discarding processes for Theorem 25

a_{1000}	l	1	2	3	4	5	6
0	U_l	$\langle \mathbf{e}_1, \mathbf{e}_2 \rangle$	$\langle \mathbf{e}_2, \mathbf{e}_3 \rangle$	$\langle \mathbf{e}_3, \mathbf{e}_4 \rangle$	$\langle \mathbf{e}_4, \mathbf{e}_1 \rangle$	$\langle \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \rangle$	$\langle \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4 \rangle$
	$\lvert \mathcal{P}_l \rvert$	19	110	136	16	7	6

This result proves the main theorem:

Theorem 26. *In dimension four there exist only six normalized diagonal polynomials.*

References

[1] G. Cantor, Ein Beitrag zur Mannigfaltigkeitslehre, J. Reine Angew. Math. (Crelle's Journal) 84 (1878) 242–258.
[2] G. Cantor, Beiträge zur Begründung der Transfiniten Mengenlehre, Math. Ann. 46 (1885) 481–512. Translated as: Contributions to the Founding of the Theory of Transfinite Numbers, Dover, New York, 1952.
[3] R. Fueter, G. Pólya, Rationale Abzählung der Gitterpunkte, Vierteljschr. Naturforsch. Gesellsch. Zürich 58 (1923) 280–386.
[4] J.S. Lew, L. Rosenberg, Polynomial indexing of integer lattice points, I and II, J. Number Theory 10 (1978) 192–214, and 215–243.
[5] J.S. Lew, Polynomial enumeration of multidimensional lattices, Math. Systems Theory 12 (1979) 253–270.
[6] J.S. Lew, L.B. Morales, A. Sanches-Flores, Diagonal polynomials for small dimension, Math. Systems Theory 29 (1996) 305–310.
[7] L.B. Morales, J.S. Lew, An enlarged family of packing polynomials on multidimensional lattices, Math. Systems Theory 29 (1996) 293–303.
[8] L.B. Morales, Diagonal polynomials and diagonal orders on multidimensional lattices, Theory Comput. Systems 30 (1997) 367–382.
[9] L.B. Morales, J.H. Arredondo, A family of $\sim e(n-1)!$ polynomial orders of N^n, Order 16 (1999) 195–206.
[10] G. Pólya, G. Szegö, Aufgaben und Lehrsätze aus der Analysis, Springer-Verlag, New York, 1964. Translated as: Problems and Theorems in Analysis, vol. 2, Springer-Verlag, New York, 1976, Problem 243.
[11] A.L. Rosenberg, Data graphs and addressing schemes, J. Comput. System Sci. 5 (1971) 193–238.
[12] A.L. Rosenberg, Storage mappings for extendible arrays, in: R.T. Yeh (Ed.), Current Trends in Programming Methodology, vol. IV, Prentice–Hall, Englewood Cliffs, NJ, 1978, pp. 263–311.
[13] A. Sanchez, A family of $(n-1)!$ diagonal polynomials orders of N^n, Order 12 (1997).
[14] C. Smoryński, Logical Number Theory I, Springer-Verlag, New York, 1991.
[15] M.A. Vsemirnov, Two elementary proofs of the Fueter–Pólya theorem on matching polynomials (in Russian), Algebra i Analiz 13 (5) (2001) 1–15. Translation in: St. Petersburg Math. J. 13 (5) (2002) 705–715.